UROPS Report

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Abstract

In this paper, we explain Goldman's parameterization of the deformation space of convex real projective structures on a pair of pants. We have also replaced some of the more abstract parts of his proof of the parameterization with more concrete arguments from projective geometry.

Contents

1 Introduction

The purpose of this paper is to explain Goldman's parameterization of the space of properly convex $\mathbb{RP}^2$ structure on a pair of pants. More precisely, let C $= \{(\lambda_1, \tau_1, \lambda_2, \tau_2, \lambda_3, \tau_3, s, t) : (\lambda_i, \tau_i \in \mathfrak{R}), s, t \in \mathbb{R}\}$. and we will parameterize the space of convex $\mathbb{RP}^2$ structures on S by C.
The strategy to do so is to construct two intermediate sets, A and B which are defined as follows.
Let $\tilde{A}$ denote the set of pairs (Ω, ρ), so that $\Omega \subset \mathbb{RP}^2$ is properly convex, $\rho : \Gamma \to PGL_3(\mathbb{R})$ is a homomorphism, and the following holds:

- $\rho(\Gamma) \subseteq Aut(\Omega)$,
- $\rho(A), \rho(B), \rho(C)$ lie in $\mathbf{Hyp}_+$,
- An axis of $\rho(A), \rho(B), \rho(C)$ lies in the boundary of Ω,
- $\rho(A^-) < \rho(A^+) < \rho(B^-) < \rho(B^+) < \rho(C^-) < \rho(C^+) < \rho(A^-)$ in the cyclic order on $\partial\Omega$.

We denote A as ${}^{\tilde{A}}\!/_{PGL_3(\mathbb{R})}$, where $g(\Omega, \rho) = (g\Omega, g\rho(\cdot)g^{-1})$ for some $g \in PGL_3(\mathbb{R})$.
$\tilde{B}$ to be the set $\{(H, \overline{A}, \overline{B}, \overline{C}) : H \subseteq \mathbb{RP}^2$ is a properly convex hexagon, $\overline{A}, \overline{B}, \overline{C} \in PGL_3(\mathbb{R})$ such that

- $\overline{A} \cdot \triangle_1 = \triangle_2, \overline{B} \cdot \triangle_2 = \triangle_3, \overline{C} \cdot \triangle_3 = \triangle_1$
- $\overline{C}\,\overline{B}\,\overline{A} = Id$
- $\overline{A}, \overline{B}, \overline{C}$ are diagonalizable with positive eigenvalues.$\}$

Define $B = {}^{\tilde{B}}\!/_{PGL_3(\mathbb{R})}$, where $g(H, \overline{A}, \overline{B}, \overline{C}) = (g \cdot H, g\overline{A}g^{-1}, g\overline{B}g^{-1}, g\overline{C}g^{-1})$ for some $g \in PGL_3(\mathbb{R})$.
By standard topological arguments, the space of convex projective structures on a pair of pants is in bijection with A. We establish a bijection between A and B, and a bijection between B and C. In Goldman's paper, his proof that A and B are in bijection uses some abstract results from the theory of geometric structures. Our proof uses more concrete arguments involving positivity of flags.
The rest of the paper is structured as follows. In Section 2, we give a brief review of real projective geometry. In particular, we define properly convex domains, as well as the cross ratio and triple ratio. Then in Section 3, we prove the Goldman parameterization.

2 The real projective plane

In this section, we define the real projective plane and explain some of its key features. We will also define the notion of a properly convex domain, and use it to define convex $\mathbb{RP}^2$ structures on a pair of pants. Finally, we will introduce two projective invariants, namely the cross ratio and the triple ratio.

2.1 $\mathbb{RP}^2$

The real projective plane $\mathbb{RP}^2$ is the space of all lines through the origin in $\mathbb{R}^3$. We can observe that for any non-zero vector v in $\mathbb{R}^3$ and for any non-zero real number k, both v and kv span the same line $\in \mathbb{R}^3$. Hence, $\mathbb{RP}^2$ can be regarded as a quotient space, ${\mathbb{R}^3\backslash\{0\}}/{\mathbb{R}^\times}$, where $\mathbb{R}^\times = \mathbb{R}\backslash\{0\}$ and $\mathbb{R}^\times$ acts on $\mathbb{R}^3$ by scaling. We used the notation [v] to represent the point $\mathbb{RP}^2$ corresponding to the line in $\mathbb{R}^3$ spanned by the vector v. In $\mathbb{RP}^2$, we have a natural notion of lines. Every line in $\mathbb{RP}^2$ is a collection of lines that lie in a plane through the origin in $\mathbb{R}^3$.
Given any space, it is a natural question to ask what is its automorphism group. We will now describe Aut($\mathbb{RP}^2$). Recall that $GL_3(\mathbb{R})$ is the group of 3×3 invertible matrices of real numbers, which acts on $\mathbb{R}^3$ by linear transformations. Let $PGL_3(\mathbb{R})$ denote the set of projective classes of matrices in $GL_3(\mathbb{R})$, i.e. $PGL_3(\mathbb{R})$ = ${GL_3(\mathbb{R})}/{\mathbb{R}^\times}$, where $\mathbb{R}^\times$ acts on $\mathbb{R}^3\backslash\{0\}$ by scaling. The linear $GL_3(\mathbb{R})$ action on $\mathbb{R}^3$ induces a natural $PGL_3(\mathbb{R})$ action on $\mathbb{RP}^2$, which we refer to as projective transformations. Also, $PGL_3(\mathbb{R})$ is isomorphic to $SL_3(\mathbb{R})$, a matrix subgroup of $GL_3(\mathbb{R})$ with determinant = 1. Explicitly, every point in $PGL_3(\mathbb{R})$ is a projective class of matrices, which has a unique representative that lies in $SL_3(\mathbb{R})$. The bijection between $PGL_3(\mathbb{R})$ simply identifies every point in $PGL_3(\mathbb{R})$ with this representative in $SL_3(\mathbb{R})$.

$\mathbb{RP}^2$ is a two-dimensional manifold, so at every point in $\mathbb{RP}^2$, it locally resembles a $\mathbb{R}^2$. We refer to these planes as affine charts.

Definition 2.1 (Affine Chart)**.** *An affine chart in $\mathbb{RP}^2$ is the complement of a line in $\mathbb{R}^3$.*

As an affine chart is $\mathbb{RP}^2\backslash$ a line $\in \mathbb{RP}^2$, we can regard that line as a plane p in $\mathbb{R}^3$. We can choose another plane p' in $\mathbb{R}^3$, whereby p needs to be parallel to p, such that it does not intersect the origin. Every line through the origin in $\mathbb{R}^3$ that does not lie in p will intersect p' at a unique point. This defines a bijection between any affine chart of $\mathbb{RP}^2$ with $\mathbb{R}^2$.

We choose an affine chart that contains the following three points, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\0\end{bmatrix}$. These points correspond to the coordinate axes in $\mathbb{R}^3$.

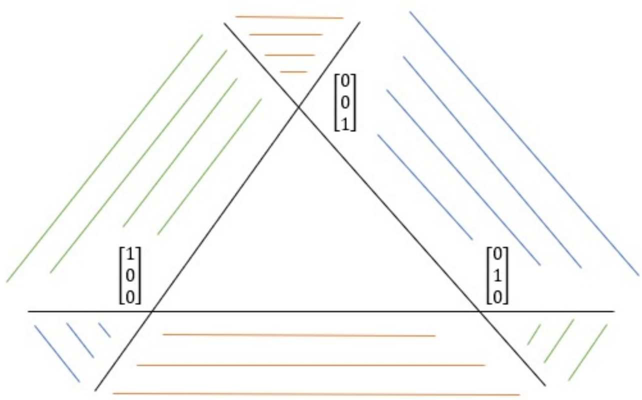

Figure 1: Geometrical interpretation of $\mathbb{RP}^2$ on an affine chart

In Figure 2, the 3 black projective lines correspond to the xy, yz and xz plane in $\mathbb{R}^3$. These projective lines split $\mathbb{RP}^2$ into 4 regions, where are represented by different colours in Figure 2.

The centre white region is

$$\{[x, y, z] \in \mathbb{RP}^2 | x > 0,\ y > 0,\ z > 0\},$$

the orange region is

$$\{[x,y,z] \in \mathbb{RP}^2 | x > 0,\ y > 0,\ z < 0\},$$

the green region is

$$\{[x,y,z] \in \mathbb{RP}^2 | x > 0,\ y < 0,\ z > 0\}$$

and the blue region is

$$\{[x,y,z] \in \mathbb{RP}^2 | x < 0,\ y > 0,\ z > 0\}$$

A projective transformation which fixes the three vertices of the white triangle, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is represented by a diagonal matrix M in $SL_3(\mathbb{R})$. If all the eigenvalues of M are positive, the transformation will leave all triangles invariant.

On the other hand, if the eigenvalues of M are not all positive, then two of the three eigenvalues must be negative, and the other has to be positive. In that case, all four regions will be interchanged.

For example, the matrix $\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$ will cause the blue region to interchange with the centre region and the green region to interchange with the orange region. The matrix $\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$ will cause the orange region to interchange with the centre region and the green region to interchange with the blue region. The matrix $\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$ will cause the green region to interchange with the centre region and the orange region to interchange with the blue region. In particular, if a projective transformation is to fix the three vertices of a triangle, and maps the triangle to itself, then it has to have only positive eigenvalues. This motivates the definition.

Definition 2.2 ($\mathbf{Hyp}_+$)**.** *Consider a matrix $A \in SL_3(\mathbb{R})$. A is said to be hyperbolic if it has 3 distinct real eigenvalues and A is positive hyperbolic if it has 3 distinct real positive eigenvalues. We denote the set of positive hyperbolic elements of $SL_3(\mathbb{R})$ by $\mathbf{Hyp}_+$.*

Recall that, h, g $\in PGL_3(\mathbb{R})$ are said to be conjugate if $\exists k \in PGL_3(\mathbb{R})$ such that $h = kgk^{-1}$. A $\in \mathbf{Hyp}_+$ is conjugate in $SL_3(\mathbb{R})$ to a diagonal matrix with positive eigenvalues.

Suppose A $\in SL_3(\mathbb{R})$. We define $\lambda(A)$ to be the real eigenvalue of A with the smallest absolute value and $\tau(A)$ to be the sum of the other two eigenvalues. Thus if A $\in \mathbf{Hyp}_+$ is conjugate to the diagonal matrix $\begin{bmatrix}\lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \nu\end{bmatrix}$ with $\lambda\mu\nu = 1$, $0 < \lambda < \mu < \nu$ then $\lambda(A) = \lambda, \tau(A) = \mu + \nu$.

Proposition 2.3 ($\mathfrak{R}$)**.** *Let $A \in \mathbf{Hyp}_+$. Then, $(\lambda(A), \tau(A)) \in \mathfrak{R}$, where $\mathfrak{R}$ is defined to be the set $\{(\lambda, \tau) \in \mathbb{R}^2 \,|\, 0 < \lambda < 1, \frac{2}{\sqrt{\lambda}} < \tau < \lambda + \lambda^{-2}\}$.*

Proof. Let A $\in \mathbf{Hyp}_+$, let $0 < \lambda < \mu < \nu$ be the eigenvalues of A. Observe that $0 < \lambda < 1$.
Since $\lambda\mu\nu = 1$, $\mu = \frac{1}{\nu\lambda}$.

$$\begin{aligned}\frac{d\tau}{d\nu} &= \frac{d(\frac{1}{\lambda\nu} + \nu)}{d\nu}\\ &= 1 - \frac{1}{\lambda\nu^2}\end{aligned}$$

Since $\lambda\mu\nu = 1 < \lambda\nu^2$, the derivative is positive. So, we can increase ν and decrease μ to increase τ and vice versa. We can obtain the supremum by approximating μ to λ and obtain the infimum by approximating μ to ν. For the supremum, μ is approximately λ and $\nu = \frac{1}{\lambda^2}$, so $\tau = \lambda + \frac{1}{\lambda^2}$. For the infimum, μ is approximately ν, so $\tau = \frac{2}{\sqrt{\lambda}}$. □

2.2 Properly convex domain

In this subsection, we want to define what a properly convex domain means. This allows us to establish a bijection between the deformation space of convex real projective structures on a pair of pants to a proper convex domain in $\mathbb{RP}^2$.

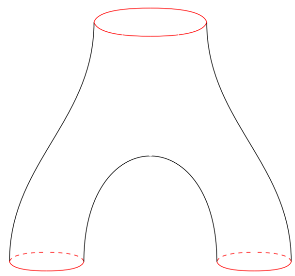

Figure 2: Geometrical interpretation of pair of pants
Taken from Wikipedia, Pair of pants (mathematics)

Definition 2.4 (Properly convex domain)**.** *A properly convex domain is an open set,* $\forall$ *a, b in the open set, there exist a projective line segment between a and b which will also lie in the domain. Furthermore, its closure lies in the affine chart.*

Suppose $\Omega \subseteq \mathbb{RP}^2$. Let $\Gamma = < A, B, C \,|\, CBA >$ where {A, B, C} is the generating set of Γ and CBA is the relation.

If A lies in $\mathbf{Hyp}_+$, A will have 3 distinct positive eigenvalues. The eigenvector with the smallest eigenvalue will correspond to the repelling fixed point of A and the eigenvector with the biggest eigenvalue will correspond to the attracting fix point of A. Any one of the two projective line segments between the attracting and repelling fix points of A is the axis of A. Let $\tilde{A}$ denote the set of pairs (Ω, ρ), so that $\Omega \subset \mathbb{RP}^2$ is properly convex, $\rho : \Gamma \to PGL_3(\mathbb{R})$ is a homomorphism, and the following holds:

- $\rho(\Gamma) \subseteq Aut(\Omega)$,
- $\rho(A), \rho(B), \rho(C)$ lie in $\mathbf{Hyp}_+$,
- An axis of $\rho(A), \rho(B), \rho(C)$ lies in the boundary of Ω,
- $\rho(A^-) < \rho(A^+) < \rho(B^-) < \rho(B^+) < \rho(C^-) < \rho(C^+) < \rho(A^-)$ in the cyclic order on $\partial\Omega$.

We denote A as $\tilde{A} / PGL_3(\mathbb{R})$, where $g(\Omega, \rho) = (g\Omega, g\rho(\cdot)g^{-1})$ for some $g \in PGL_3(\mathbb{R})$.

Remark 2.5. {*Convex projective structures on pair of pants*} $\leftrightarrow$ *A by standard topological arguments.*

2.3 General position

Given 4 points in $\mathbb{RP}^2$, these 4 points are considered to be in general position if no 3 or more points lie on a projective line. This allows us to state Lemma 2.6, which we will use later.

Lemma 2.6. *Let A, B, C, D and A', B', C', D' be two quadruple of points in general position. There exist a unique projective transformation in* $PGL_3(\mathbb{R})$ *that sends a, b, c, d to a', b', c', d' respectively.*

Proof. The strategy is to find a projective transformation that sends A, B, C, D to A", B", C" and D" respectively, where a", b", c", d" are the respective vector representatives of A", B", C", D" and a" $= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, b" $= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, c" $= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and d" $= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. For the same reasons, there is a projective transformation that sends A', B', C' and D' to A", B", C" and D" respectively, so there will be a composite mapping from A, B, C and D to A', B', C' and D' respectively.
Let a, b, c, d be the respective vector representatives of A, B, C, D. Since a, b, c form a basis of $\mathbb{R}^3$, d is a linear combination of a, b, c, i.e. $d = \alpha a + \beta b + \gamma c$, where $\alpha, \beta, \gamma \in \mathbb{R}$.

Let $\bar{g} \in GL_3(\mathbb{R})$, be the linear map defined by $\bar{g} \cdot a = \begin{pmatrix} \frac{1}{\alpha} \\ 0 \\ 0 \end{pmatrix}$, $\bar{g} \cdot b = \begin{pmatrix} 0 \\ \frac{1}{\beta} \\ 0 \end{pmatrix}$ and $\bar{g} \cdot c = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\gamma} \end{pmatrix}$ and define $g \in PGL_3(\mathbb{R})$ to be the projective class of matrices containing $\bar{g}$. It is clear that g·A = A", g·B = B" and g·C = C".

Observe that $\bar{g} \cdot d = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, so $g \cdot$ D = D".

Next, we show that g is unique.

Suppose g' is a projective transformation that has a representative $\overline{g'}$ in $GL_3(\mathbb{R})$, so $g' \cdot a = \begin{pmatrix} k' \\ 0 \\ 0 \end{pmatrix}$, $g' \cdot b = \begin{pmatrix} 0 \\ l' \\ 0 \end{pmatrix}$ and $g' \cdot c = \begin{pmatrix} 0 \\ 0 \\ m' \end{pmatrix}$, $g' \cdot d = \begin{pmatrix} r' \\ r' \\ r' \end{pmatrix}$. Then, $\begin{bmatrix} \alpha k \\ \beta l \\ \gamma m \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha k' \\ \beta l' \\ \gamma m' \end{bmatrix}$.

This implies that $k = k' = \frac{1}{\alpha}, l = l' = \frac{1}{\beta}, m = m' = \frac{1}{\gamma}$. Therefore, the mapping is unique. □

2.4 Dual space

In this subsection, we want to define the dual of $\mathbb{RP}^2$. With the notion of the dual space, we can construct the triple ratio.

The dual of $\mathbb{R}^3$ is denoted as $(\mathbb{R}^3)^*$. $(\mathbb{R}^3)^*$ is defined to be {linear maps from $\mathbb{R}^3 \to \mathbb{R}$}. The dimension of $(\mathbb{R}^3)^*$ is 3 and every non-zero element in the dual space has nullity 2. Hence, its nullspace is a plane in $\mathbb{R}^3$ through the origin. We can observe that $(\mathbb{R}^3)^*$ is a three-dimensional vector space as it satisfies the following properties: Suppose $L_1, L_2 \in (\mathbb{R}^3)^*$, where $L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto ax+by+cz$, $L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto a'x+b'y+c'z$, then

- $L_1 + L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto (a+a')x + (b+b')y + (c+c')z$
- $kL_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto kax + kby + kcz$

Given that the basis for $\mathbb{R}^3$ is e_1, e_2 and e_3, the dual basis for $(\mathbb{R}^3)^*$ would be e_1^*, e_2^* and e_3^*, where

$$e_1^* : e_1 \to 1, e_2 \to 0, e_3 \to 0$$
$$e_2^* : e_1 \to 0, e_2 \to 1, e_3 \to 0$$
$$e_3^* : e_1 \to 0, e_2 \to 0, e_3 \to 1$$

Similarly, the dual of $\mathbb{RP}^2$ is denoted as $(\mathbb{RP}^2)^*$. $(\mathbb{RP}^2)^*$ is defined to be ${}^{(\mathbb{R}^3)^* \backslash \{0\}}\!/_{\mathbb{R}^\times}$. Since a plane in $\mathbb{R}^3$ is equivalent to lines in $\mathbb{RP}^2$, $(\mathbb{RP}^2)^*$ is equivalent to lines in $\mathbb{RP}^2$. As $\mathbb{RP}^2$ is represented as $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\}$, where every point in $(\mathbb{RP}^2)^*$ can be represented as {[a, b, c]}

Remark 2.7. *There is a bijection between points in $(\mathbb{RP}^2)^*$ and lines in $\mathbb{RP}^2$, where {points in $(\mathbb{RP}^2)^*$} $\leftrightarrow$ {lines in $\mathbb{RP}^2$}. Also, there is a bijection between lines in $(\mathbb{RP}^2)^*$ and points in $\mathbb{RP}^2$, where {lines in $(\mathbb{RP}^2)^*$} $\leftrightarrow$ {points in $\mathbb{RP}^2$}.*

2.5 Cross Ratio

In this subsection, we want to define the cross ratio and explain some of its key features. This is an important tool in projective geometry to help us establish the parametrization of convex real projective structures on a pair of pants.

The value of the cross ratio is a number that associated with 4 points on a projective line.

Definition 2.8. *Let A, B, C, D be four pairwise distinct points in $\mathbb{RP}^2$ that lie in a projective line l. Choose an affine chart containing A, B, C, D, and choose an orientation on l. In $\mathbb{R}^3$, A,B,C,D correspond to four lines through the origin that lie in a plane, which corresponds to l. The choice of affine chart corresponds to a plane P in $\mathbb{R}^3$ that does not intersect the origin, but intersects the four lines. Choose vector representatives a, b, c, d of A, B, C, D whose forward endpoint lies in the plane P. Then define $\mathcal{C}(A, B, C, D) = \frac{\overline{ac}\,\overline{db}}{\overline{ab}\,\overline{dc}}$. The four quantities $\overline{ac}, \overline{db}, \overline{ab}, \overline{dc}$ are signed distances measured in the Euclidean metric in $\mathbb{R}^3$, where the sign is determined by the direction of the projective line l.*

Remark 2.9. *While the sign of each of the four terms depend on the direction of the projective line, the value of cross ratio does not.*

Lemma 2.10. *The value of the cross ratio does not depend on our choice of affine chart.*

Proof. Given 4 collinear points A, B, C, D on an affine chart. We can regard them as lines in $\mathbb{R}^3$, which cuts through the origin. The affine chart contains A, B, C, D if and only if as a plane in $\mathbb{R}^3$, it intersects the four lines in $\mathbb{R}^3$ corresponding to A, B, C, D. As shown in Figure 3, we can choose the red or blue or green line as our affine chart. With every selection of our affine chart, we will have 4 points of intersections, which we can take as the vector representatives of A, B, C, D used in the definition of the cross ratio above. For our proof, we shall use the red and blue line. Let o be the origin,

$$\begin{aligned}
\frac{\overline{ac}\,\overline{db}}{\overline{ab}\,\overline{dc}} &= \frac{\frac{\overline{ac}}{\overline{oa}}}{\frac{\overline{bc}}{\overline{ob}}}\frac{\overline{oa}}{\overline{ob}}\frac{\frac{\overline{bd}}{\overline{ob}}}{\frac{\overline{ad}}{\overline{oa}}}\frac{\overline{ob}}{\overline{oa}} \qquad (*) \\
&= \frac{\frac{\sin aoc}{\sin aco}}{\frac{\sin boc}{\sin bco}}\frac{\frac{\sin bod}{\sin bdo}}{\frac{\sin aod}{\sin ado}} \\
&= \frac{\sin aoc}{\sin boc}\frac{\sin bod}{\sin aod} \qquad (\sin(\theta) = \sin(\pi - \theta)) \\
&= \frac{\sin a'oc'}{\sin b'oc'}\frac{\sin b'od'}{\sin a'od'} \\
&= \frac{\overline{a'c'}\,\overline{d'b'}}{\overline{a'b'}\,\overline{d'c'}}
\end{aligned}$$

$(*)$ There would be an even number of negative sine value and positive sine value, hence the negative sign will cancel one another out, thus the cross ratio will be the same. Without loss of generality, we remove all minus sign. $\square$

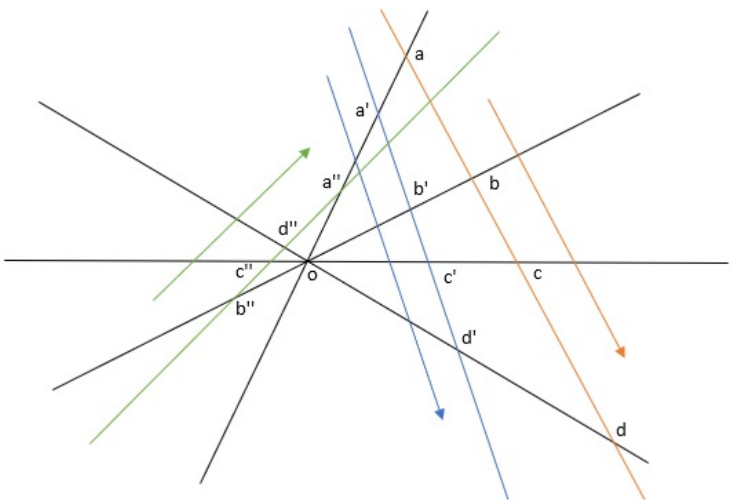

Figure 3: Different permutation of cross ratio

Different permutations of the 4 unique points can produce different values of the cross ratio. There are $4! = 24$ ways to permute. Properties of cross ratio are as follows:

1. $\mathcal{C}(A,B,C,D) = \mathcal{C}(D,C,B,A)$
2. $\mathcal{C}(A,B,C,D) = \frac{1}{\mathcal{C}(A,C,B,D)}$
3. $\mathcal{C}(A,B,C,D) = 1 - \mathcal{C}(B,A,C,D)$

$\mathcal{C}(A,B,C,D) = \mathcal{C}(B,A,D,C) = \mathcal{C}(C,D,A,B) = \mathcal{C}(D,C,B,A) = x$
$\mathcal{C}(A,C,B,D) = \mathcal{C}(C,A,D,B) = \mathcal{C}(B,D,A,C) = \mathcal{C}(D,B,C,A) = \frac{1}{x}$
$\mathcal{C}(A,B,D,C) = \mathcal{C}(B,A,C,D) = \mathcal{C}(D,C,A,B) = \mathcal{C}(C,D,B,A) = 1 - x$
$\mathcal{C}(A,C,D,B) = \mathcal{C}(C,A,B,D) = \mathcal{C}(D,B,A,C) = \mathcal{C}(B,D,C,A) = \frac{1-x}{x}$
$\mathcal{C}(A,D,B,C) = \mathcal{C}(D,A,C,B) = \mathcal{C}(B,C,A,D) = \mathcal{C}(C,B,D,A) = \frac{1}{1-x}$
$\mathcal{C}(A,D,C,B) = \mathcal{C}(D,A,B,C) = \mathcal{C}(C,B,A,D) = \mathcal{C}(B,C,D,A) = \frac{x}{1-x}$

Theorem 2.11. *Given two different projective lines in $\mathbb{RP}^2$, we can choose 2 sets of 4 collinear points, A, B, C, D and A', B', C', D' on each projective line. The cross ratio $\mathcal{C}(A, B, C, D) = \mathcal{C}(A', B', C', D')$ iff $\exists g \in PGL_3(\mathbb{R})$ such that $gA = A'$, $gB = B'$, $gC = C'$ and $gD = D'$*

Proof. Let a and d be the respective vector representative of A and D and a' and d' be the respective vector representative of A' and D'. Let b and c be the vector representatives of B and C, whereby b and c can be expressed as a linear combination of a and d. Also, let b' and c' be the respective position vector of B' and C', whereby b' and c' can be expressed as a linear combination of a' and d'.

($\Leftarrow$) Let a = $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, d = $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$. Since A, B, C, D are collinear, b = $\begin{pmatrix} (1-t)d_1 + ta_1 \\ (1-t)d_2 + ta_2 \\ (1-t)d_3 + ta_3 \end{pmatrix}$ and c = $\begin{pmatrix} (1-s)d_1 + sa_1 \\ (1-s)d_2 + sa_2 \\ (1-s)d_3 + sa_3 \end{pmatrix}$, for some s, t $\in \mathbb{R}$.

$$\overline{ac} = \sqrt{((1-s)(a_1 - d_1))^2 + ((1-s)(a_2 - d_2))^2 + ((1-s)(a_3 - d_3))^2} = (1-s)\overline{ad}$$

Similarly, $\overline{ab} = (1-t)\overline{ad}, \overline{dc} = s\overline{ad}, \overline{db} = t\overline{ad}$. So, the cross ratio $\mathcal{C}(A, B, C, D) = \frac{\overline{ac}\,\overline{db}}{\overline{ab}\,\overline{dc}} = \frac{(1-s)t}{(1-t)s}$.

Since A', B', C', D' are collinear, b' = $(1-t')$d' + ta' and c' = $(1-s')$d' + sa', for some $s', t' \in \mathbb{R}$.
Let $g \in PGL_3(\mathbb{R})$ be a projective transformation such that $g\cdot$a = ja' and $g\cdot$d = kd' for some $j, k \in \mathbb{R}$.
$g\cdot$c = $m(1-s')$d' + sa' = $(1-s)\,k$d' + $s\,j$a', where $m \in \mathbb{R}$.
$\implies m(1-s') = (1-s)k,\ ms' = sj \implies (1-s) = \frac{m}{k}(1-s'),\ s = \frac{m}{j}(s')$
$g\cdot$b = $n(1-t')$d' + ta' = $(1-t)\,k$d' + $t\,j$a', where $n \in \mathbb{R}$.
$\implies n(1-t') = (1-t)k,\ nt' = tj \implies (1-t) = \frac{n}{k}(1-t'),\ t = \frac{n}{j}(t')$
Since the cross ratio of $\mathcal{C}(A', B', C', D') = \frac{\overline{a'c'}\,\overline{d'b'}}{\overline{a'b'}\,\overline{d'c'}} = \frac{(1-s')t'}{(1-t')s'}$, therefore

$$\mathcal{C}(A, B, C, D) = \frac{\overline{ac}\,\overline{db}}{\overline{ab}\,\overline{dc}} = \frac{(1-s)t}{(1-t)s} = \frac{\frac{m}{k}(1-s')\frac{n}{j}(t')}{\frac{n}{k}(1-t')\frac{m}{j}(s')} = \frac{\overline{a'c'}\,\overline{d'b'}}{\overline{a'b'}\,\overline{d'c'}} = \frac{(1-s')t'}{(1-t')s'} = \mathcal{C}(A', B', C', D')$$

($\Rightarrow$) Let a, b, c, d be the vector representatives of A, B, C, D that lie on the same projective line. As before, there is some non-zero real numbers s,t so that $b = ta + (1-t)d, c = sa + (1-s)d$. The cross ratio $\mathcal{C}(A, B, C, D) = \frac{\overline{ac}\,\overline{db}}{\overline{ab}\,\overline{dc}} = \frac{(1-s)t}{(1-t)s}$.

The strategy is to find a projective transformation that sends A, B, C, D to A", B", C" and D", with vector representatives a" = $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, b" = $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$, c" = $\begin{pmatrix} h \\ 1-h \\ 0 \end{pmatrix}$ and d" = $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. For the same reasons, A', B', C' and D' can also be mapped to A", B", C" and D" via a projective transformation, so there will be a composite mapping from A, B, C and D to A', B', C' and D' respectively.

We can construct a $\bar{g} \in SL_3(\mathbb{R})$, where $\bar{g} \cdot a = \begin{pmatrix} \frac{1-t}{t} \\ 0 \\ 0 \end{pmatrix}$ and $\bar{g} \cdot d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, since a and d are linearly independent.

The cross ratio $\mathcal{C}(A", B", C", D") = \frac{\overline{a"c"}\,\overline{d"b"}}{\overline{a"b"}\,\overline{d"c"}} = \frac{(1-h)}{h}$.

We assume that $\mathcal{C}(A, B, C, D) = \mathcal{C}(A", B", C", D")$.

$\bar{g}\cdot$b = $\bar{g}(ta + (1-t)d) = \begin{pmatrix} (1-t) \\ (1-t) \\ 0 \end{pmatrix}$, since b" = $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$, and $\bar{g}\cdot$c = $g(sa + (1-s)d) = \begin{pmatrix} s\frac{(1-t)}{t} \\ (1-s) \\ 0 \end{pmatrix}$.
Let $g \in PGL_3(R)$ be the projective class of matrices containing $\bar{g} \in SL_3(R)$. g is the required projective transformation, where $g\cdot$A = A", $g\cdot$B = B", $g\cdot$D = D" and $g\cdot$C = C". Hence, C" = $\begin{bmatrix} \frac{s}{1-s}\frac{(1-t)}{t} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{h}{1-h} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} h \\ 1-h \\ 0 \end{bmatrix}$. ∎

Remark 2.12. *This defines $\Omega^* \subset (\mathbb{RP}^2)^*$ by $\{\Omega^* = [a, b, c] : \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq 0, \forall \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \Omega\}$.*

With the bijection between $\mathbb{RP}^2$ and $(\mathbb{RP}^2)^*$, 4 points and 1 line contained in $\mathbb{RP}^2$ can be expressed as 4 lines intersecting at 1 point in $(\mathbb{RP}^2)^*$ and vice versa. The cross ratio for both geometrical structures is the same.

We can define a cross ratio on a pair of points in $\mathbb{RP}^2$ and a pair of lines in $\mathbb{RP}^2$ such that the two points do not lie in the two lines.

Definition 2.13. *Let A, B be two points in $\mathbb{RP}^2$, and X, Y be two projective lines in $\mathbb{RP}^2$ that do not contain A, B. Let v, w be vector representatives of A, B respectively, and let α, β be linear functionals whose kernels are X, Y respectively. Then define $\mathcal{C}(X, A, B, Y) = \frac{\alpha(w)\beta(v)}{\alpha(v)\beta(w)}$.*

This does not depend on the choice of representatives α, β for the linear functionals and v, w for the vector representatives and is a projective invariant.

Lemma 2.14. *Let A, B, C, D be four points in $\mathbb{RP}^2$ that lie in a projective line. Let L_A be a projective line that contains A, but does not contain B, C, D. Similarly, let L_D be a projective line that contains D but does not contain A, B, C. Then $\mathcal{C}(A, B, C, D) = \mathcal{C}(L_A, B, C, L_D)$.*

Proof. Based on Theorem 2.11, we can normalise the vector representatives to those shown in Figure 4. By using the the 4 vector representatives, $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}s\\1-s\\0\end{pmatrix}, \begin{pmatrix}t\\1-t\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\mathcal{C}(A, B, C, D) = \frac{(1-s)t}{(1-t)s}$. By using the vectors $\begin{pmatrix}s\\1-s\\0\end{pmatrix}, \begin{pmatrix}t\\1-t\\0\end{pmatrix}$ and the linear maps (1, 0, 0) and (0, 1, 0), we can compute that $\mathcal{C}(L_A, B, C, L_D) = \frac{(1-s)t}{(1-t)s}$. Hence, we show that $\mathcal{C}(A, B, C, D) = \mathcal{C}(L_A, B, C, L_D)$.

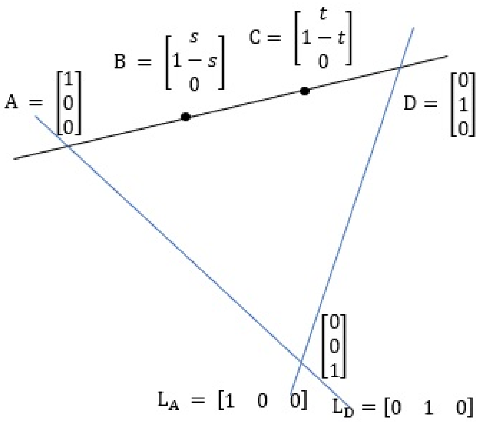

Figure 4: Illustration for cross ratio of 2 linear functional and 2 vector representative

□

2.6 Triple ratio

In this subsection, we want to define the triple ratio and explain some of its key features. This is an important tool in projective geometry that will help us establish the parametrization of convex real projective structures on a pair of pairs. The triple ratio is similar to cross ratio, whereby the triple ratio is a function that takes in 3 flags and outputs a unique number.

Definition 2.15 (Flag)**.** *Flag*$(\mathbb{R}^3) = \{(p, l) \in \mathbb{RP}^2 \times (\mathbb{RP}^2)^* : l(p) = 0\}$*. A flag F can be represented as a tuple, $(F^{(1)}, F^{(2)})$, where $F^{(1)}$ is a point on an affine chart in $\mathbb{RP}^2$ and $F^{(2)}$ is a projective line, which contains $F^{(1)}$, on an affine chart in $\mathbb{RP}^2$.*

Let (F_1, F_2, F_3) be three flags such that $F_i^{(1)}$ does not lie in $F_j^{(2)}\ \forall\, i \neq j$, as shown in Figure 5.

Definition 2.16 (Triple ratio)**.** *For i=1,2,3, let v_i be a vector representative of $F_i^{(1)}$ and let α_i be a linear map whose kernel is $F_i^{(2)}$. The triple ratio of (F_1, F_2, F_3) is given by $\mathcal{T}(F_1, F_2, F_3) = \frac{\alpha_1(v_2)\alpha_2(v_3)\alpha_3(v_1)}{\alpha_1(v_3)\alpha_3(v_2)\alpha_2(v_1)}$.*

Triple ratio does not depend on the choice of representatives α_i for $F_i^{(2)}$ and v_i for $F_i^{(1)}$ and is a projective invariant.

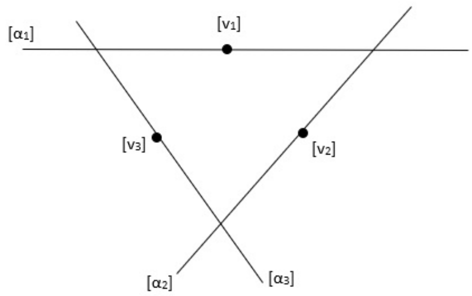

Figure 5: Illustration for triple ratio

A key tool that we will use to prove Goldman's parameterization is the notion of a positive n-tuple of flags.

Definition 2.17. *Let $(F_1, ..., F_n)$ be a cyclically ordered n-tuple of flags in general position, for $n \in \mathbb{Z}$. $(F_1, F_2, ..., F_n)$ is said to be positive if $\forall F_i, F_j, F_k$ where $i < j < k < i$, the value of the triple ratio, $\mathcal{T}(F_i, F_j, F_k) > 0$ and $\forall F_i, F_j, F_k, F_l$ where $i < j < k < l < i$, the cross ratio, $\mathcal{C}(F_i^{(2)}, F_j^{(1)}, F_l^{(1)}, F_k^{(1)} + F_i^{(1)}) < 0$.*

Lemma 2.18. *The value of the triple ratio is positive $\Leftrightarrow \exists$ connected component of $\mathbb{RP}^2 \backslash (F_i^{(2)} \cup F_j^{(2)} \cup F_k^{(2)})$ that contains $F_i^{(1)}, F_j^{(1)}, F_k^{(1)}$ in its boundary.*

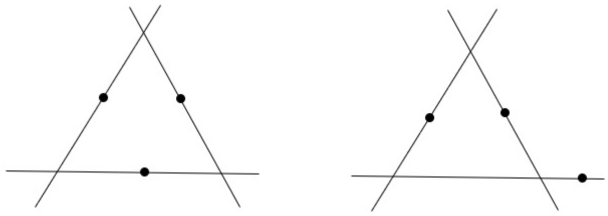

Figure 6: Cases where triple ratio is positive or negative

Proof. We normalise the flags on the projective lines and set the linear operator, as shown in Figure 12. As the bottom projective line may vary, the dotted lines denote the value of α according to the projective line. In Figure 7, the bottom projective line lies within the region of $0 < \alpha < \infty$. The value of the triple ratio is $\dfrac{\begin{pmatrix}\alpha & 1 & -1-\alpha\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}}{\begin{pmatrix}\alpha & 1 & -1-\alpha\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}} = \alpha$. So the triple ratio is positive. Furthermore, there exists a connected component of $\mathbb{RP}^2 \backslash (F_i^{(2)} \cup F_j^{(2)} \cup F_k^{(2)})$ that contains $F_i^{(1)}, F_j^{(1)}, F_k^{(1)}$ in its boundary. However, if that projective line were to lie within the region $-\infty < \alpha < 0$, the value of the triple ratio would be negative and there does not exist a connected component of $\mathbb{RP}^2 \backslash (F_i^{(2)} \cup F_j^{(2)} \cup F_k^{(2)})$ that contains $F_i^{(1)}, F_j^{(1)}, F_k^{(1)}$ in its boundary.

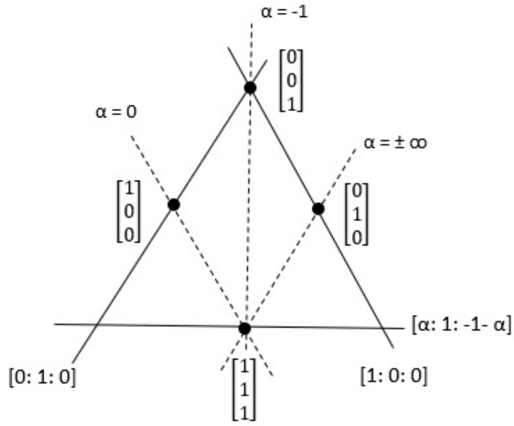

Figure 7: Case where triple ratio is positive

□

3 Properly convex $\mathbb{RP}^2$ structure on pair of pants

In this section, the goal is to parametrize the space of properly convex $\mathbb{RP}^2$ structures on a pair of pants. Recall that we previously defined the set $\tilde{A}$ and A as follows.
Let $\tilde{A}$ denote the set of pairs (Ω, ρ), so that $\Omega \subset \mathbb{RP}^2$ is properly convex, $\rho : \Gamma \to PGL_3(\mathbb{R})$ is a homomorphism, and the following holds:

- $\rho(\Gamma) \subseteq Aut(\Omega)$,
- $\rho(A), \rho(B), \rho(C)$ lie in $\mathbf{Hyp}_+$,
- An axis of $\rho(A), \rho(B), \rho(C)$ lies in the boundary of Ω,
- $\rho(A^-) < \rho(A^+) < \rho(B^-) < \rho(B^+) < \rho(C^-) < \rho(C^+) < \rho(A^-)$ in the cyclic order on $\partial\Omega$.

We denote A as ${}^{\tilde{A}}\!/_{PGL_3(\mathbb{R})}$, where $g(\Omega, \rho) = (g\Omega, g\rho(\cdot)g^{-1})$ for some $g \in PGL_3(\mathbb{R})$.
Also, define $\tilde{B}$ to be the set $\{(H, \overline{A}, \overline{B}, \overline{C}) : H \subseteq \mathbb{RP}^2$ is a properly convex hexagon, $\overline{A}, \overline{B}, \overline{C} \in PGL_3(\mathbb{R})$ such that

- $\overline{A} \cdot \triangle_1 = \triangle_2, \overline{B} \cdot \triangle_2 = \triangle_3, \overline{C} \cdot \triangle_3 = \triangle_1$
- $\overline{C}\,\overline{B}\,\overline{A} = Id$
- $\overline{A}, \overline{B}, \overline{C}$ are diagonalizable with positive eigenvalues.$\}$

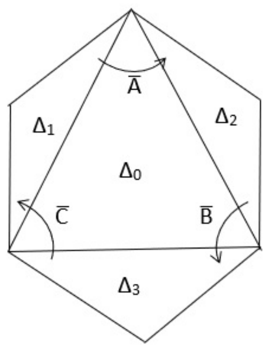

Figure 8: Illustration of 4 triangles on hexagon

Define $B = {}^{\tilde{B}}\!/_{PGL_3(\mathbb{R})}$, where $g(H, \overline{A}, \overline{B}, \overline{C}) = (g \cdot H, g\overline{A}g^{-1}, g\overline{B}g^{-1}, g\overline{C}g^{-1})$ for some $g \in PGL_3(\mathbb{R})$.

Finally, define C to be $\{(\lambda_1, \tau_1, \lambda_2, \tau_2, \lambda_3, \tau_3, s, t) : (\lambda_i, \tau_i \in \mathfrak{R}), s, t \in \mathbb{R}\}$.

Recall that the deformation space of convex real projective structures on a pair of pants is identified with A. Thus, to prove Goldman's parameterization, we need to establish a bijection between A and C. This will be done in two steps. In Section 3.1, we give a bijection betwee B and C, and in Section 3.2, we give a bijection between A and B.

3.1 Parameterization of convex hexagons

In this subsection, we establish a bijection between B and C.

We assign homogeneous coordinates to the hexagon, as shown in Figure 9.

The cross ratio is unique to the same 4 lines. Consider the point $\begin{bmatrix}0\\1\\0\end{bmatrix}$, we can calculate $\mathcal{C}(L_1, p_1, p_2, L_2)$, by taking p_1 as $\begin{bmatrix}1\\0\\0\end{bmatrix}$, p_2 as $\begin{bmatrix}0\\0\\1\end{bmatrix}$, the two blue lines L_1, L_2 as shown in Figure 9. We can find 2 cross ratio in a similar fashion. The calculation for one cross ratio will be shown for reference.
First, the intersections of the blue lines (extension of projective lines) and the orange line are found to be

$\begin{bmatrix} a_3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ c_1 \end{bmatrix}$. As the 4 points are collinear, $l \begin{pmatrix} 1 \\ 0 \\ c_1 \end{pmatrix}$ = (1-s) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ + s $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $m \begin{pmatrix} a_3 \\ 0 \\ 1 \end{pmatrix}$ = (1-t) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ + t $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for s, t, m and l $\in \mathbb{R}$. The cross ratio $\rho_2 = \frac{s(1-t)}{t(1-s)}$. From the linear combination of 4 collinear points, $c_1 = \frac{s}{1-s}$ and $a_3 = \frac{1-t}{t}$. So

$$\rho_2 = c_1 a_3 \tag{1}$$

Similarly, the other cross ratios can be computed to be

$$\rho_1 = b_3 c_2 \tag{2}$$

$$\rho_3 = a_2 b_1 \tag{3}$$

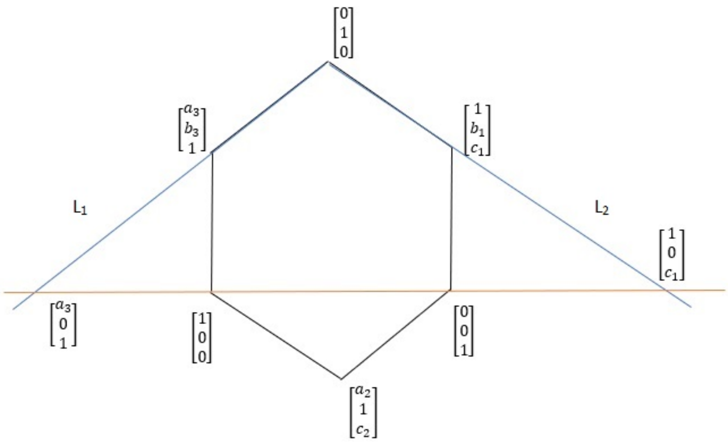

Figure 9: Finding cross ratio on hexagon

Now, we want to choose a representative hexagon in each equivalence class. As we have proven that we can map any 4 points in general position to any other 4 points in general position, we can fix the coordinates of the hexagon to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ by projective transformation, as shown in Figure 10.

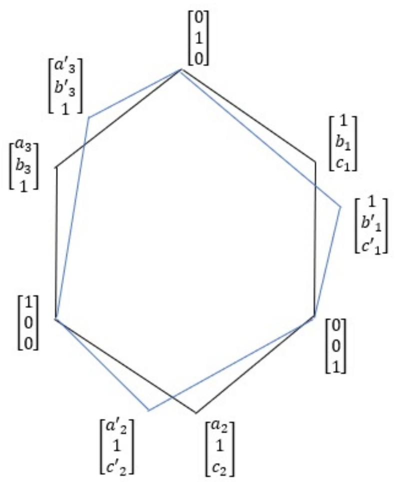

Figure 10: Geometrical interpretation of 2 hexagons

Remark 3.1. *If the blue hexagon $\sim$ the black hexagon, $\exists \gamma \in PGL_3(\mathbb{R})$ such that*

$$\gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \gamma \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \begin{bmatrix} a_3' \\ b_3' \\ -1 \end{bmatrix} = \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix}, \gamma \begin{bmatrix} a_2' \\ -1 \\ c_2' \end{bmatrix} = \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix}, \gamma \begin{bmatrix} -1 \\ b_1' \\ c_1' \end{bmatrix} = \begin{bmatrix} -1 \\ b_1 \\ c_1 \end{bmatrix}$$

From the first 3 equalities, it can be deduced that $\gamma = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$.

Using the fourth equality as an example,

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \begin{bmatrix} a_3' \\ b_3' \\ -1 \end{bmatrix} = \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix} \implies \begin{bmatrix} \lambda a_3' \\ \mu b_3' \\ -\nu \end{bmatrix} = \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix}$$

By factoring out $\frac{1}{\lambda}$,

$$\frac{\mu}{\lambda} b_1' = b_1, \; \frac{\nu}{\lambda} c_1' = c_1$$

Similarly,

$$\frac{\lambda}{\mu} a_2' = a_2, \; \frac{\nu}{\mu} c_2' = c_2, \; \frac{\lambda}{\nu} a_3' = a_3, \; \frac{\mu}{\nu} b_3' = b_3$$

With this, one can verify that the two invariants are also projective invariants.

$$\sigma_1 = a_2 b_3 c_1 \tag{4}$$

$$\sigma_2 = a_3 b_1 c_2 \tag{5}$$

There is a relation where

$$\rho_1 \rho_2 \rho_3 = \sigma_1 \sigma_2$$

As we have proven that we can map any 4 points in general position to any other 4 points in general position. This means that we can normalize any 4 points in general position. As we have normalized 3 points previously, we can choose one more point to normalize. In this case, we choose to normalize our fourth point to $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. Therefore, $a_3 = 2$ and $b_3 = 2$.

This implies that $a_2 = \frac{\sigma_1}{\rho_2}$, $b_1 = \frac{\rho_3 \rho_2}{\sigma_1}$, $c_1 = \frac{\rho_2}{2}$, $c_2 = \frac{\rho_1}{2}$. Now, we define a parameter t, where $t = \frac{\sigma_1}{\rho_2} > 0$. This means that $a_3 = 2$, $b_3 = 2$, $a_2 = t$, $b_1 = \frac{\rho_3}{t}$, $c_1 = \frac{\rho_2}{2}$, $c_2 = \frac{\rho_1}{2}$.

According to Figure 4, matrix A maps vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix} \mapsto \beta_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \gamma_1 \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix}$$

Matrix B maps vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \alpha_2 \begin{bmatrix} -1 \\ b_1 \\ c_1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \beta_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix} \mapsto \gamma_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Matrix C maps vectors

$$\begin{bmatrix} -1 \\ b_1 \\ c_1 \end{bmatrix} \mapsto \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \beta_3 \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \gamma_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Remark 3.2. *Given the 3 transformations A, B, C, we can deduce that matrix A takes the form* $\begin{bmatrix} \alpha_1 & \epsilon_1 & \gamma_1 a_3 \\ 0 & \epsilon_2 & \gamma_1 b_3 \\ 0 & \epsilon_3 & -\gamma_1 \end{bmatrix}$. *Notice that A maps* $\begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix} \mapsto \beta_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$*, so* $\epsilon_3 = -\gamma_1 c_2$*,* $\epsilon_1 = \alpha_2 a_2 + \gamma_1 a_3 c_2$ *and* $\epsilon_2 = -\beta_1 + \gamma_1 b_3 c_2$*. Matrix B and C can deduced in a similar fashion.*

Hence, A $= \begin{bmatrix} \alpha_1 & \alpha_2 a_2 + \gamma_1 a_3 c_2 & \gamma_1 a_3 \\ 0 & -\beta_1 + \gamma_1 b_3 c_2 & \gamma_1 b_3 \\ 0 & -\gamma_1 c_2 & -\gamma_1 \end{bmatrix}$, B $= \begin{bmatrix} -\alpha_2 & 0 & -\alpha_2 a_3 \\ \alpha_2 b_1 & \beta_2 & \beta_2 b_3 + \alpha_2 a_3 b_1 \\ \alpha_2 c_1 & 0 & -\gamma_2 + \alpha_2 a_3 c_1 \end{bmatrix}$ and C $= \begin{bmatrix} -\alpha_3 + \beta_3 a_2 b_1 & \beta_3 a_2 & 0 \\ -\beta_3 b_1 & -\beta_3 & 0 \\ \gamma_3 c_1 + \beta_3 b_1 c_2 & \beta_3 c_2 & \gamma_3 \end{bmatrix}$,

where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3 > 0$.

Also,

$$detA = \alpha_1 \beta_1 \gamma_1 = 1 \tag{6}$$

$$detB = \alpha_2 \beta_2 \gamma_2 = 1 \tag{7}$$

$$detC = \alpha_3 \beta_3 \gamma_3 = 1 \tag{8}$$

It follows that CBA maps

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \alpha_1 \alpha_2 \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix} = \beta_1 \beta_2 \beta_3 \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \gamma_1 \gamma_2 \gamma_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, CBA = Id iff

$$\alpha_1 \alpha_2 \alpha_3 = \beta_1 \beta_2 \beta_3 = \gamma_1 \gamma_2 \gamma_3 = 1 \tag{9}$$

We can then find the characteristic polynomial of A, B, C. The characteristic polynomial of A is $(x - \alpha_1)(x^2 + (\beta_1 + \gamma_1 - \gamma_1 b_3 c_2)x)$. Therefore,

$$\lambda(A) = \lambda_1 = \alpha_1, \; \tau(A) = \tau_1 = -\beta_1 + \gamma_1(\rho_1 - 1) \tag{10}$$

Similarly, we can find the characteristic polynomial of B and C to derive $\lambda(B), \tau(B), \lambda(C), \tau(C)$.

$$\lambda(B) = \lambda_2 = \beta_2, \; \tau(B) = \tau_2 = -\gamma_2 + \alpha_2(\rho_2 - 1) \tag{11}$$

$$\lambda(C) = \lambda_3 = \gamma_3, \; \tau(C) = \tau_3 = -\alpha_3 + \beta_3(\rho_3 - 1) \tag{12}$$

Remark 3.3. *It is difficult to calculate the eigenvalues directly, hence it is more appropriate to use the λ and τ function to make the calculations easier.*

Define s by the equation $\log(\alpha_2) = \frac{1}{2}\log(\frac{\lambda_3}{\lambda_1\lambda_2}) - \log(s)$. Define t = a_2. Assign to the point $(H, \overline{A}, \overline{B}, \overline{C})$ in B the point $(\lambda(A), \tau(A), \lambda(B), \tau(B), \lambda(C), \tau(C)$, s, t). This defines a map from B to C.

Now, we will show that you can invert this map.
We can then take the logarithms of equations 9, 10, 11 and 12. By the way how we defined s, we get

$$\log(\gamma_2) = \frac{1}{2}\log(\frac{\lambda_1}{\lambda_3\lambda_2}) - \log(s)$$

$$\log(\alpha_3) = \frac{1}{2}\log(\frac{\lambda_2}{\lambda_3\lambda_1}) - \log(s)$$

$$\log(\beta_3) = \frac{1}{2}\log(\frac{\lambda_1}{\lambda_3\lambda_2}) + \log(s)$$

$$\log(\beta_1) = \frac{1}{2}\log(\frac{\lambda_3}{\lambda_1\lambda_2}) - \log(s)$$

$$\log(\gamma_1) = \frac{1}{2}\log(\frac{\lambda_2}{\lambda_3\lambda_1}) - \log(s)$$

This allows us to solve,

$$\alpha_1 = \lambda_1, \; \alpha_2 = \sqrt{\frac{\lambda_3}{\lambda_2\lambda_1}} s^{-1}, \; \alpha_3 = \sqrt{\frac{\lambda_2}{\lambda_3\lambda_1}} s$$

$$\beta_1 = \sqrt{\frac{\lambda_3}{\lambda_2\lambda_1}} s, \; \beta_2 = \lambda_2, \; \beta_3 = \sqrt{\frac{\lambda_1}{\lambda_2\lambda_3}} s^{-1}$$

$$\gamma_1 = \sqrt{\frac{\lambda_2}{\lambda_3\lambda_1}} s^{-1}, \; \gamma_2 = \sqrt{\frac{\lambda_1}{\lambda_2\lambda_3}} s, \; \gamma_3 = \lambda_3$$

Using the τ function in equation 10, 11, 12, we can get the following ρ expression by substituting the corresponding variables.

$$\rho_1 = 1 + \sqrt{\frac{\lambda_1\lambda_3}{\lambda_2}} \tau_1 s + \frac{\lambda_3}{\lambda 2} s^2$$

$$\rho_2 = 1 + \sqrt{\frac{\lambda_1\lambda_2}{\lambda_3}}\tau_2 s + \frac{\lambda_1}{\lambda 3}s^2$$

$$\rho_3 = 1 + \sqrt{\frac{\lambda_2\lambda_3}{\lambda_1}}\tau_3 s + \frac{\lambda_2}{\lambda 1}s^2$$

Hence, the coordinates a_2, a_3, b_1, b_3, c_1, c_2 are

$$a_2 = t,\ a_3 = 2$$

$$b_1 = \frac{1}{2} + \sqrt{\frac{\lambda_2\lambda_3}{\lambda_1}}\tau_3\frac{s}{t} + \frac{\lambda_2}{\lambda_1}\frac{s^2}{t},\ b_3 = 2$$

$$c_1 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\lambda_2\lambda_1}{\lambda_3}}\tau_2 s + \frac{\lambda_1}{2\lambda_3}s^2,\ c_2 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\lambda_3\lambda_1}{\lambda_2}}\tau_1 s + \frac{\lambda_3}{2\lambda_2}s^2$$

This calculation shows that we can recover $(H, \overline{A}, \overline{B}, \overline{C})$ from the data $(\lambda(A), \tau(A), \lambda(B), \tau(B), \lambda(C), \tau(C)$, s, t). In particular, the map from B to C defined above is a bijection.

3.2 Bijection from deformation space to convex hexagons

In this subsection, we want to establish a bijection between A and B. Observe that this is equivalent to establishing a bijection between $\tilde{A}$ and $\tilde{B}$.
We shall construct 2 functions Φ, Ψ, where $\Phi : \tilde{A} \to \tilde{B}$ and $\Psi : \tilde{B} \to \tilde{A}$ such that $\Phi\Psi = Id = \Psi\Phi$. This will imply that Φ is a bijection.

First, we construct the function Φ. Given the pair $(\Omega, \rho) \in \tilde{A}$, we can form a triangle in Ω by connecting the repelling fixed points, $\rho(A^-), \rho(B^-), \rho(C^-)$.

Remark 3.4. *The 3 respective points of the hexagon which remains invariant when $\overline{A}, \overline{B}, \overline{C}$ act on them are actually the repelling fixed points. According to equation 10, 11 and 12, they correspond to the smallest eigenvalues of $\overline{A}, \overline{B}, \overline{C}$. Hence, we use the repelling fixed points to construct the triangle.*

Lemma 3.5. *$\rho(A^+) < \rho(C^-) < \rho(B^-) < \rho(AC^+) < \rho(AC^-) < \rho(A^+)$ in the clockwise cyclic order on $\partial\Omega$. Furthermore, the line segment between $\rho(AC^-)$ and $\rho(AC^+)$ lies in $\partial\Omega$.*

Proof. We can transform the boundary by a transformation and connect the 3 new repelling fixed points, $\rho(AC^-), \rho(CB^-), \rho(BA^-)$ to form 3 other triangles, as shown in Figure 11. As $\rho(A^+)$ is the attracting fixed point, $\rho(A)$ will transform $\rho(C^-)$ toward $\rho(A^+)$, so $\rho(C^-) < \rho(AC^-) < \rho(A^+) < \rho(C^-)$ in the clockwise cyclic order on $\partial\Omega$. Notice that $\rho(CBA) = Id$, so $\rho(A) = \rho(B^{-1})\rho(C^{-1})$. Hence, $\rho(B^-) < \rho(AC^-) < \rho(C^+) < \rho(B^-)$ in the clockwise cyclic order on $\partial\Omega$. For the same reasons, $\rho(B^-) < \rho(AC^+) < \rho(A^+) < \rho(B^-)$ in the clockwise cyclic order on $\partial\Omega$. Since the $\rho(A)$ action on $\partial\Omega$ is continuous and bijective, so the lemma holds. □

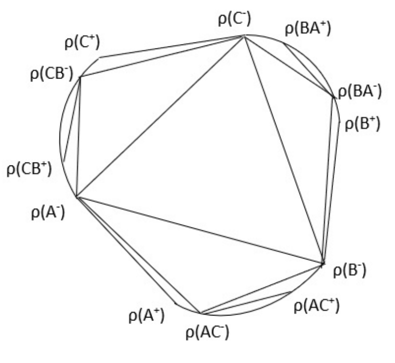

Figure 11: Construction of function Φ

By taking $\rho(C^-), \rho(BA^-), \rho(B^-), \rho(AC^-), \rho(A^-), \rho(CB^-)$ as the vertices, we can obtain a hexagon.

Hence, we have constructed a map from $(\Omega, \rho) \to (H, \overline{A}, \overline{B}, \overline{C})$ such that $\overline{A} = \rho(A), \overline{B} = \rho(B), \overline{C} = \rho(C)$.

Lemma 3.6. *The value of the cross ratio is negative iff $F_j^{(1)}, F_l^{(1)}$ lies in different connected components of $F_i^{(2)}, F_k^{(1)} + F_i^{(1)}$.*

Proof. Using Lemma 2.14 and property of projective invariance of cross ratio, we can normalise $F_j^{(1)}, F_l^{(1)}, F_i^{(2)}, F_k^{(1)} + F_i^{(1)}$ and get 4 points that lie in a projective line. Since the direction has to be chosen when calculating the cross ratio, there will be an odd number of negative distances and positive distances, according to Subsection 2.5 Cross Ratio. Hence, the value of the cross ratio is negative. □

Theorem 3.7. *For $n \in \mathbb{Z}, (F_1, ..., F_n)$ is positive and $F_1, ..., F_n$ are in general position iff one of the component of $\mathbb{RP}^2 \backslash (F_1^{(2)} \cup F_j^{(2)} \cup ... \cup F_n^{(2)})$ is a properly convex n-sided polygon that contains $F_1^{(1)}, F_2^{(1)}, ..., F_n^{(1)}$ in its boundary in this order.*

Proof. (⇒)
We prove this by induction. Let P(n) be the statement that if $(F_1, ..., F_n)$ is positive and $F_1, ..., F_n$ are in general position then one of the component of $\mathbb{RP}^2 \backslash (F_1^{(2)} \cup F_j^{(2)} \cup ... \cup F_n^{(2)})$ is a properly convex n-sided polygon that contains $F_1^{(1)}, F_2^{(1)}, ..., F_n^{(1)}$ in its boundary in this order for n ≥ 3.
For P(3), it is a consequence of Lemma 2.18.

We assume that P(k) holds, for some integer k ≥ 4.

We can check whether P(4) holds. First, we can normalise the vector representatives as shown in Figure 8. We can check that the value of the triple ratio of $F_i^{(1)}, F_j^{(1)}, F_k^{(1)}$ is positive, so $F_j^{(2)}$ would only contain $F_j^{(1)}$. Since $(F_i^{(1)}, F_j^{(1)}, F_k^{(1)}, F_l^{(1)})$ is positive, the value of the cross ratio should be negative. By taking $F_i^{(2)}$ and $F_k^{(1)} + F_i^{(1)}$, $F_j^{(1)}$ and $F_l^{(1)}$ should lie in different connected component. Also, if we take $F_k^{(2)}$ and $F_k^{(1)} + F_i^{(1)}$, $F_j^{(1)}$ and $F_l^{(1)}$ should lie in different connected component. Hence, we can deduce the position of $F_l^{(1)}$, which is shown in Figure 12. We can also check that the value of the triple ratio of $F_i^{(1)}, F_l^{(1)}, F_k^{(1)}$ is positive, hence $F_l^{(2)}$ would only contain $F_l^{(1)}$. There will then be a properly convex 4-sided polygon.

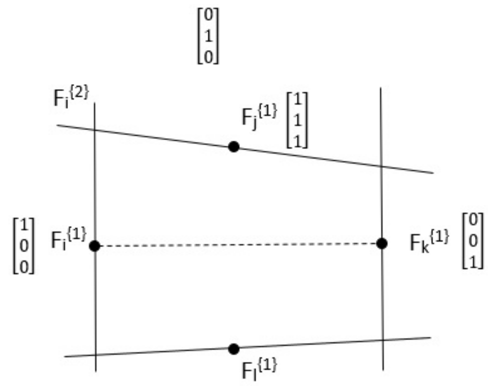

Figure 12: Illustration for base case

We can use similar argument to prove the nth case.

We can observe that if $(F_1, ..., F_n, F_{n+1})$ is positive, then $(F_1, ..., F_n)$ is positive. Thus, we may apply the inductive hypothesis to deduce that the n+1 sided polygon is a properly convex domain. So given a n-sided properly convex polygon, we can construct a new side as shown in Figure 13. By Lemma 3.6, we deduce the position of $F_{n+1}^{(1)}$ as $\mathcal{C}(F_1^{(2)}, F_2^{(1)}, F_{n+1}^{(1)}, F_1^{(1)} + F_n^{(1)})$ and $\mathcal{C}(F_n^{(2)}, F_2^{(1)}, F_{n+1}^{(1)}, F_1^{(1)} + F_n^{(1)})$ are negative. Hence, it can be deduced that $F_{n+1}^{(1)}$ is located on the red line, as shown in Figure 13. Also, by Lemma 2.18, we can deduce the position of $F_{n+1}^{(2)}$ as $\mathcal{T}(F_1, F_n, F_{n+1})$ is positive. Hence, it can be deduced that $F_{n+1}^{(2)}$ will intersect $F_1^{(2)}$ and $F_n^{(2)}$, as shown in Figure 13. Therefore, it can be deduced that the new domain would be a n+1 sided properly convex domain. It can be seen in Figure 13, where we removed the shaded region.

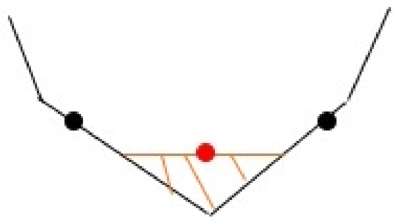

Figure 13: Illustration for inductive step

(⇐)

Given a n-sided polygon, we can choose a point that lies in the interior of every edge of the polygon. We will denote these points as our flags in a cylical order. We can then choose any 4 flags, F_i, F_j, F_k, F_l in the cylical order. By Lemma 3.6, $\mathcal{C}(F_i^{(2)}, F_j^{(1)}, F_l^{(1)}, F_k^{(1)} + F_i^{(1)})$ and $\mathcal{C}(F_k^{(2)}, F_j^{(1)}, F_l^{(1)}, F_k^{(1)} + F_i^{(1)})$ will be negative. Also, we can choose any 3 flags F_i, F_j, F_k in the given cylical order. By Lemma 2.18, $\mathcal{T}(F_i, F_j, F_k)$ will be positive. Hence, this means that $F_1, ..., F_n$ on the edge of the polygon are in general position and $(F_1, ..., F_n)$ is positive. □

Corollary 3.8. *Suppose $F_1, ..., F_n$ are in general position. Let $M \subseteq \mathbb{R}^2$ be a n-sided polygon. We enumerate the vertices of M by $p_1, ..., p_n$ in this cyclic order. We define a map that send the set of vertices in M to Flag($\mathbb{R}^n$). We can choose a triangulation of M such that the vertices of the triangles are also vertices of M. $(F_1, ..., F_n)$ is positive ⇔*

- *If p_i, p_j, p_k are vertices of a triangle in M, then $\mathcal{T}(F_i, F_j, F_k) > 0$,*
- *If p_i, p_j, p_k and p_l, p_k, p_i are adjacent triangle in M, then $\mathcal{C}(F_i^{(2)}, F_j^{(1)}, F_l^{(1)}, F_k^{(1)} + F_i^{(1)}) < 0$.*

Remark 3.9. *If a domain is proper, any smaller region within the domain is still proper.*

With Theorem 3.8, we can then construct the function Ψ and a nested sequence of hexagons, where
$H_1 \subset H_2 \subset H_3 ....$

Our H_1 is the starting hexagon. Given the H_i, we can define H_{i+1} as follows: ∀ boundary edge h of H_i, $\exists g_h \in PGL_3(\mathbb{R})$ such that $g_h \cdot T_1$ or $g_h \cdot T_2$ is a triangle that has h as an edge but does not lie in H_i. So H_{i+1} is $\bigcup_{h\, edges\, in\, H_i} g \cdot T \cup H_i$. By Corollary 3.9, we set the vertices of H_i as flags and choose a triangulation on H_i such that the triangle created by transforming T_1 or T_2 will add on to the triangulation, as shown in Figure 14.
We can observe that the quadruple with vertices (F_1, F_2, F_n, F_{n+1}) are transformed from another quadruple with vertices F_i, F_j, F_k, F_l within H_i. Since H_i is properly convex, so $\mathcal{C}(F_i^{(2)}, F_j^{(1)}, F_l^{(1)}, F_k^{(1)} + F_i^{(1)})$ and $\mathcal{C}(F_k^{(2)}, F_j^{(1)}, F_l^{(1)}, F_k^{(1)} + F_i^{(1)})$ is negative. As cross ratio is projective invariant, $\mathcal{C}(F_1^{(2)}, F_2^{(1)}, F_{n+1}^{(1)}, F_1^{(1)} + F_n^{(1)})$ and $\mathcal{C}(F_n^{(2)}, F_2^{(1)}, F_{n+1}^{(1)}, F_1^{(1)} + F_n^{(1)})$ must be negative. By Lemma 3.6, we can deduce that the vertex of the new triangle must lie within $F_1^{(2)}$ and $F_n^{(2)}$, as shown in Figure 14.
We can observe that the triangle with vertices (F_1, F_n, F_{n+1}) are transformed from another triangle with vertices F_i, F_j, F_k within H_i. Since H_i is properly convex, so $\mathcal{T}(F_i, F_j, F_k)$ is positive. As triple ratio is projective invariant, F_1, F_n, F_{n+1} must be negative. By Lemma 2.18, we can deduce that $F_{n+1}^{(2)}$ will intersect $F_1^{(2)}$ and $F_n^{(2)}$, as shown in Figure 14.

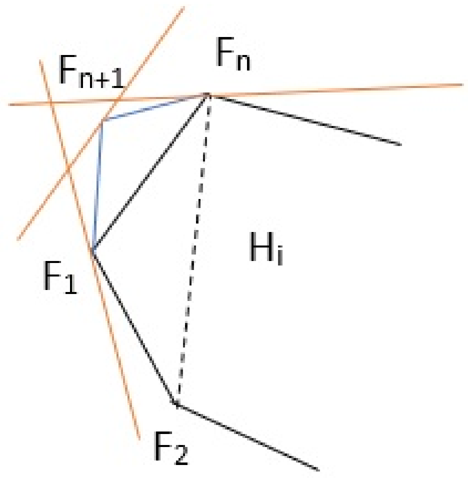

Figure 14: Construction of function Ψ

Hence, after a transformation, our H_2 is a 12-sided properly convex polygon. After the second transformation, the our H_3 is 24-sided properly convex polygon. We can observe that the inner polygon is expanding and the outer polygon is shrinking, to a point where both will converge to the same polygon. The shrinking outer polygon explains the properness of the domain. As the initial outer polygon is proper, a smaller outer polygon would also be proper. The expanding inner polygon explains the convexity of the polygons. With the choice of any two points in the domain, we can keep applying the projective transformation until both points lie in the union of the polygons. This means that the domain is convex. We can observe points in the inner hexagon will approach the attracting fixed points $\rho(A^+), \rho(B^+), \rho(C^+)$ after many transformations. With the convexity and the properness of the domain, the line segment $\rho(A^-A^+), \rho(B^-B^+)$ and $\rho(C^-C^+)$ would lie on the boundary of the domain.

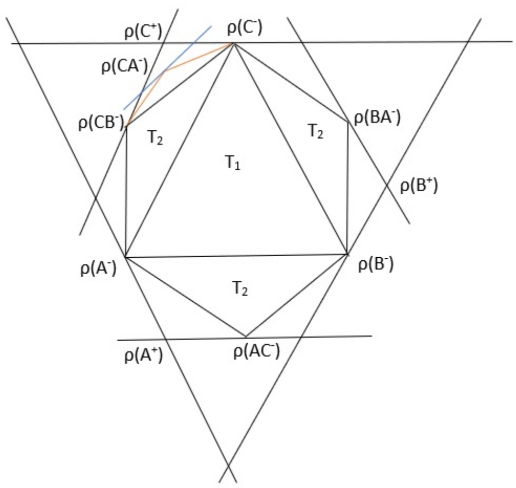

Figure 15: Construction of function Ψ

Therefore, we have established the function Φ and Ψ, thus found a bijection between $\tilde{A}$ and $\tilde{B}$. Hence, we have found a parametrization on convex real projective structure on a pair of pants.